1 Material indices for elastic beams with differing constraints (Figure 1). Start each of the four parts of this problem by listing the function, the objective, the constraints and the free variable. You will need the equations for the deflection of a cantilever beam with a square cross-section *t* x *t*. The two that matter are that for the deflection δ of a beam of length *L* under an end load:

$$\delta = \frac{FL^3}{3EI}$$

and that for the deflection of a beam under a distributed load *f* per unit length:

$$\delta = \frac{1}{8}\frac{fL^4}{EI}$$

where *I* is given in Table 2. For a self-loaded beam $f = \rho A g$ where ρ is the density of the material of the beam, *A* its cross-sectional area and *g* the acceleration due to gravity.

(a) Show that the best material for a cantilever beam of given length *L* and given (i.e. fixed) square cross-section (*t x t*) that will deflect least under a given end load *F* is that with the largest value of the index $M = E$, where *E* is Young's modulus (neglect self-weight). (Figure 1a.)

(b) Show that the best material choice for a cantilever beam of given length *L* and with a given section (*t* x *t*) that will deflect least under its own self-weight is that with the largest value of $M = E/\rho$, where ρ is the density. (Figure 1b.)

(c) Show that the material index for the lightest cantilever beam of length *L* and square section (not given, i.e., the area is a free variable) that will not deflect by more than δ under its own weight is $M = E / \rho^2$. (Figure 1c.)

(d) Show that the lightest cantilever beam of length *L* and square section (area free) that will not deflect by more than δ under an end load *F* is that made of the material with the largest value of $M = E^{1/2} / \rho$ (neglect self weight). (Figure 1d.)

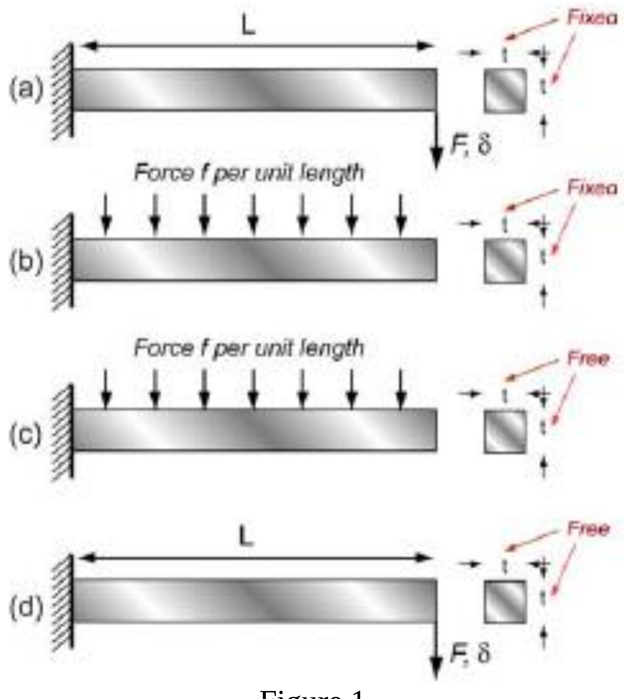
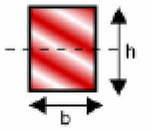
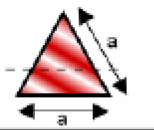
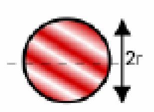
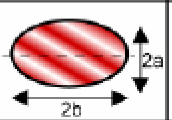
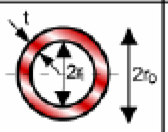
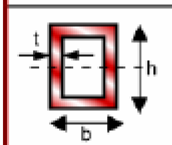
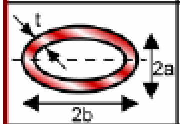
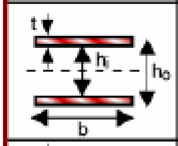
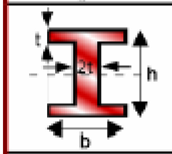
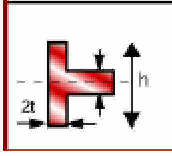


Figure 1

Functions required:

2. Moments of sections.

Section shape	Area A m	Moment I m^4	Moment K m^4	Moment Z m^4	Moment Q m^4	Moment Z_p m^4
	bh	$\frac{bh^3}{12}$	$\frac{bh^3}{3}(1-0.58\frac{b}{h})$ $(h>b)$	$\frac{bh^2}{6}$	$\frac{b^2h^2}{(3h+1.8b)}$ $(h>b)$	$\frac{bh^2}{4}$
	$\frac{\sqrt{3}}{4}a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{\sqrt{3}a^4}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$	$\frac{3a^3}{64}$
	πr^2	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}r^3$	$\frac{\pi}{2}r^3$	$\frac{\pi}{3}r^3$
	πab	$\frac{\pi}{4}a^3b$	$\frac{\pi a^3b^3}{(a^2+b^2)}$	$\frac{\pi}{4}a^2b$	$\frac{\pi}{2}a^2b$ $(a<b)$	$\frac{\pi}{3}a^2b$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3t$	$\frac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3t$	$\frac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2t$	$\frac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2pr^2t$	$\frac{\pi}{3}(r_o^3-r_i^3)$ $\approx \pi r^2t$
	$2t(h+b)$ $(h,b>>t)$	$\frac{1}{6}h^3t(1+3\frac{b}{h})$	$\frac{2tb^2h^2}{(h+b)}(1-\frac{t}{h})^4$	$\frac{1}{3}h^2t(1+3\frac{b}{h})$	$2tbh(1-\frac{t}{h})^2$	$bht(1+\frac{h}{2b})$
	$\pi(a+b)t$ $(a,b>>t)$	$\frac{\pi}{4}a^3t(1+\frac{3b}{a})$	$\frac{4\pi(ab)^{5/2}t}{(a^2+b^2)}$	$\frac{\pi}{4}a^2t(1+\frac{3b}{a})$	$2\pi t(a^3b)^{1/2}$ $(b>a)$	$\pi abt(2+\frac{a}{b})$
	$b(h_o-h_i)$ $\approx 2bt$ $(h,b>>t)$	$\frac{b}{12}(h_o^3-h_i^3)$ $\approx \frac{1}{2}bth_o^2$	--	$\frac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	--	$\frac{b}{4}(h_o^2-h_i^2)$ $\approx bth_o$
	$2t(h+b)$ $(h,b>>t)$	$\frac{1}{6}h^3t(1+3\frac{b}{h})$	$\frac{2}{3}bt^3(1+4\frac{h}{b})$	$\frac{1}{3}h^2t(1+3\frac{b}{h})$	$\frac{2}{3}bt^2(1+4\frac{h}{b})$	$bht(1+\frac{h}{2b})$
	$2t(h+b)$ $(h,b>>t)$	$\frac{t}{6}(h^3+4bt^2)$	$\frac{t^3}{3}(8b+h)$	$\frac{t}{3h}(h^3+4bt^2)$	$\frac{t^2}{3}(8b+h)$	$\frac{th^2}{2}\{1+\frac{2t(b-2t)}{h^2}\}$

Solution

Exercise 1:

The model. The point of this problem is that the material index depends on the mode of loading, on the geometric constraints and on the design goal.

(a) The table lists the design requirements for part (a) of the problem.

Function	• *End-loaded cantilever beam*
Constraints	• *Length L specified* • *Section t x t specified* • *End load F specified*
Objective	• *Minimize the deflection, δ*
Free variables	• *Choice of material only*

The objective function is an equation for the deflection of the beam. An end-load F produces a deflection δ of

$$\delta = \frac{FL^3}{3EI}$$

where E is the modulus of the beam material and $I = t^4 / 12$ is the second moment of the area, so that the deflection becomes

$$\delta = 4\frac{FL^3}{t^4}\left(\frac{1}{E}\right)$$

The magnitude of the load F and the dimensions L and t are all given. The deflection δ is minimized by maximizing:

$$M_1 = E$$

(b) The design requirements for part (b) are listed below:

Function	• *Self-loaded cantilever beam*
Constraints	• *Length L specified* • *Section t x t specified*
Objective	• *Minimize the deflection, δ*
Free variables	• *Choice of material only*

The beam carries a distributed load, f per unit length, where

$$f = \rho\ g\ t^2$$

where ρ is the density of the beam material and g is the acceleration due to gravity. Such a load produces a deflection

$$\delta = \frac{3}{2}\frac{fL^4}{Et^4} = \frac{3}{2}\frac{gL^4}{t^2}\left(\frac{\rho}{E}\right)$$

(the objective function). As before, t and L are given. The deflection is minimized by maximizing

$$M_2 = E/\rho$$

(c) The design requirements for part (c) are listed below

Function	• *Self-loaded cantilever beam*
Constraints	• *Length L specified* • *Maximum deflection, δ , specified*
Objective	• *Minimize the mass, m*
Free variables	• *Choice of material only* • *Section area A = t^2*

The beam deflects under its own weight but now the section can be varied to reduce the weight provided the deflection does not exceed δ, as in the figure. The objective function (the quantity to be minimized) is the mass *m* of the beam

$$m = t^2 L \rho$$

Substituting for *t* (the free variable) from the second equation into the first, gives

$$m = \frac{3}{2}\frac{gL^5}{\delta}\left(\frac{\rho^2}{E}\right)$$

The quantities *L* and δ are given. The mass is minimized by maximizing

$$M_3 = E/\rho^2$$

(d) The design requirements for part (d) are listed below

Function	• *End-loaded cantilever beam*
Constraints	• *Length L specified* • *Maximum deflection, δ , specified* • *End-load F specified*
Objective	• *Minimize the mass, m*
Free variables	• *Choice of material only* • *Section area A = t^2*

The section is square, but the dimension *t* is free. The objective function is

$$m = t^2 L \rho$$

The deflection is, as in part (a)

$$\delta = 4\frac{FL^3}{t^4}\left(\frac{1}{E}\right)$$

Using this to eliminate the free variable, *t*, gives

$$m = 2\left(\frac{FL^5}{\delta}\right)^{1/2}\left(\frac{\rho}{E^{1/2}}\right)$$

The quantities *F*, δ and *L* are given. The mass is minimized by maximizing

$$M_4 = E^{1/2}/\rho$$

From a selection standpoint, M_3 and M_4 are equivalent.

Lecture : Material Selection, Exercise

The selection. Applying the three indices to the CES Edu Level 1 or 2 database gives the top-ranked candidates listed below

Index Material choice	Material choice
High $M_1 = E$	• Metals: tungsten alloys, steels. • Ceramics: SiC, Si_3N_4, B_4C and AlN, but of course all are brittle.
High $M_2 = E/\rho$	• Metals: aluminum, magnesium, titanium alloys and steels all have almost the same value of E/ρ • Composites: CFRP • Ceramics SiC, Si_3N_4, B_4C and AlN
High $M_4 = E^{1/2}/\rho$	• Metals: aluminum and magnesium alloys superior to all other metals. • Composites: CFRP excels • Ceramics: SiC, Si_3N_4, B_4C and AlN